

EIGENVALUES FOR A NONLOCAL PSEUDO  $p$ -LAPLACIAN

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ABSTRACT. In this paper we study the eigenvalue problems for a nonlocal operator of order  $s$  that is analogous to the local pseudo  $p$ -Laplacian. We show that there is a sequence of eigenvalues  $\lambda_n \rightarrow \infty$  and that the first one is positive, simple, isolated and has a positive and bounded associated eigenfunction. For the first eigenvalue we also analyze the limits as  $p \rightarrow \infty$  (obtaining a limit nonlocal eigenvalue problem analogous to the pseudo infinity Laplacian) and as  $s \rightarrow 1^-$  (obtaining the first eigenvalue for a local operator of  $p$ -Laplacian type). To perform this study we have to introduce anisotropic fractional Sobolev spaces and prove some of their properties.

## 1. INTRODUCTION

Our main goal is to introduce a nonlocal operator that is a nonlocal analogous to the local pseudo  $p$ -Laplacian,  $\Delta_{p,x}u + \Delta_{p,y}u$  (here the subindexes  $x$  and  $y$  denote differentiation with respect to the  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$  variables respectively). The local pseudo  $p$ -Laplacian appears naturally when one considers critical points of the functional  $F(u) = \int_{\Omega} |\nabla_x u|^p + |\nabla_y u|^p dx dy$ . See [5, 14, 25, 33, 34]. On the other hand, recently, it was introduced a nonlocal  $p$ -Laplacian that is given by

$$(-\Delta)_p^s v(x) = 2 \text{ P.V. } \int_{\mathbb{R}^k} \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y))}{|x - y|^{k+ps}} dx,$$

the symbol P.V. stands for the principal value of the integral. We will omit it in what follows. For references involving this kind of operator we refer to [9, 16, 18, 23, 24, 26, 29, 30, 32, 31] and references therein.

Here, we introduce the following nonlocal operator that we will call the nonlocal pseudo  $p$ -Laplacian,

$$\begin{aligned} \mathcal{L}_{s,p}(u)(x, y) := & 2 \int_{\mathbb{R}^n} \frac{|u(x, y) - u(z, y)|^{p-2} (u(x, y) - u(z, y))}{|x - z|^{n+sp}} dz \\ & + 2 \int_{\mathbb{R}^m} \frac{|u(x, y) - u(x, w)|^{p-2} (u(x, y) - u(x, w))}{|y - w|^{m+sp}} dw. \end{aligned}$$

The natural space to consider when one deals with the operator  $\mathcal{L}_{s,p}$  is given by

$$\mathcal{W}^{s,p}(\mathbb{R}^{n+m}) := \left\{ u \in L^p(\mathbb{R}^{n+m}) : [u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p < \infty \right\},$$

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where for  $p < +\infty$ ,

$$[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p := \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x,y) - u(z,y)|^p}{|x-z|^{n+sp}} dz dx dy \\ + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|u(x,y) - u(x,w)|^p}{|y-w|^{m+sp}} dw dx dy$$

and for  $p = +\infty$ ,

$$[u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} := \max \left\{ \sup \left\{ \frac{|u(x,y) - u(z,y)|}{|x-z|^s} : (x,y) \neq (z,y) \right\} ; \right. \\ \left. \sup \left\{ \frac{|u(x,y) - u(x,w)|}{|y-w|^s} : (x,y) \neq (x,w) \right\} \right\}.$$

In this paper, we deal with the eigenvalue problem for this operator, that is, given a bounded domain  $\Omega$  we look for pairs  $(\lambda, u)$  such that  $\lambda \in \mathbb{R}$  and  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$  are such that  $u$  is a weak solution of

$$\begin{cases} \mathcal{L}_{s,p} u(x,y) = \lambda |u(x,y)|^{p-2} u(x,y) & \text{in } \Omega, \\ u(x,y) = 0 & \text{in } \Omega^c = \mathbb{R}^{n+m} \setminus \Omega. \end{cases}$$

Here  $\widetilde{\mathcal{W}}^{s,p}(\Omega) = \{u \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}) : u \equiv 0 \text{ in } \Omega^c\}$ . We will study the Dirichlet problem for this operator in a companion paper.

We impose the following assumptions on the data:

- A1.  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^{n+m}$ ;
- A2.  $s \in (0, 1)$ , and  $p \in (1, +\infty)$ .

Under these conditions we have the following result.

**Theorem 1.1.** *There exists a sequence of eigenvalues  $\lambda_n$  such that  $\lambda_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Moreover, every eigenfunction is in  $L^\infty(\mathbb{R}^{n+m})$ . The first eigenvalue (the smallest eigenvalue) is given by*

$$\lambda_1(s,p) := \inf \left\{ \frac{[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} : u \in \widetilde{\mathcal{W}}^{s,p}(\Omega), u \not\equiv 0 \right\}.$$

*This eigenvalue  $\lambda_1(s,p)$  is simple, isolated and an associated eigenfunction is strictly positive (or negative) in  $\Omega$ .*

Next, we analyze the limit as  $s \rightarrow 1^-$  of the first eigenvalue obtaining that there is a limit that is the first eigenvalue of a local operator that involve two  $p$ -Laplacians (one in the  $x$  variables and another one in  $y$  variables).

**Theorem 1.2.** *Let  $\Omega$  is bounded domain in  $\mathbb{R}^{n+m}$  with smooth boundary, and fix  $p \in (1, \infty)$ . Then*

$$(1.1) \quad \lim_{s \rightarrow 1^-} (1-s)\lambda_1(s,p) = \lambda_1(1,p) \\ := \inf \left\{ \frac{K_{n,p} \|\nabla_x u\|_{L^p(\Omega)}^p + K_{m,p} \|\nabla_y u\|_{L^p(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p} : u \in W_0^{1,p}(\Omega), u \not\equiv 0 \right\},$$

*where the constant  $K_{n,p} > 0$  depends only on  $n$  and  $p$ , while  $K_{m,p} > 0$  depends only on  $m$  and  $p$ .*

Observe that the limit value,  $\lambda_1(1, p)$ , is the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -K_{n,p}\Delta_{p,x}u - K_{m,p}\Delta_{p,y}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Concerning the limit as  $p \rightarrow \infty$  (for a fixed  $s$ ) for the first eigenvalue we have the following result.

**Theorem 1.3.** *It holds that*

$$\lim_{p \rightarrow \infty} [\lambda_1(s, p)]^{1/p} = \Lambda_\infty(s)$$

where

$$\Lambda_\infty(s) := \inf \{ [u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} : u \in \mathcal{W}^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ in } \Omega^c \}.$$

In addition, the eigenfunctions  $u_p$  normalized by  $\|u_p\|_{L^p(\Omega)} = 1$  converge along subsequences  $p_n \rightarrow \infty$  uniformly to a continuous limit  $u_\infty$ , that is a nontrivial viscosity solution to

$$\begin{cases} \max\{A; C\} = \max\{-B; -D; \Lambda_\infty(s)u\} & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

with

$$\begin{aligned} A &= \sup_w \frac{u(x, w) - u(x, y)}{|y - w|^s}, & B &= \inf_w \frac{u(x, w) - u(x, y)}{|y - w|^s}, \\ C &= \sup_z \frac{u(z, y) - u(x, y)}{|x - z|^s}, & D &= \inf_z \frac{u(z, y) - u(x, y)}{|x - z|^s}. \end{aligned}$$

We can give a simple geometric characterization of the limit value  $\Lambda_\infty(s)$ , this value is related to the maximum distance (measured in a way that involves the exponent  $s$ , see below) from one point  $(x, y) \in \Omega$  to the boundary. In fact,

$$\Lambda_\infty(s) = \frac{1}{\max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x - z|^s + |y - w|^s)}.$$

That the limit equation is verified in the viscosity sense and involve quotients of the form  $\frac{u(x,w)-u(x,y)}{|y-w|^s}$  is not surprising. In fact, viscosity solutions provide the right framework to deal with limits of  $p$ -Laplacians as  $p \rightarrow \infty$ , see [4, 6, 27], and quotients like the one mentioned above appeared in other related limits, see [12, 23, 29]. What is remarkable in the limit equation is that it involves the limit value  $\Lambda_\infty(s)$  and that the quotients that appear have perfectly identified the two groups of variables that are present in the fractional pseudo  $p$ -Laplacian that we introduced here.

Our results say that we can take the limits as  $s \rightarrow 1^-$  and as  $p \rightarrow \infty$  in the first eigenvalue. With the above notations we have the following commutative diagram

$$\begin{array}{ccc} ((1-s)\lambda_1(s, p))^{1/p} & \xrightarrow{s \rightarrow 1^-} & (\lambda_1(1, p))^{1/p} \\ p \rightarrow \infty \downarrow & & \downarrow p \rightarrow \infty \\ \Lambda_\infty(s) & \xrightarrow{s \rightarrow 1^-} & \Lambda_\infty. \end{array}$$

Here

$$\Lambda_\infty := \frac{1}{\max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x-z| + |y-w|)}.$$

The limit

$$\lim_{p \rightarrow \infty} (\lambda_1(1, p))^{1/p} = \Lambda_\infty$$

can be obtained as in [27] using the variational characterization of  $\lambda_1(1, p)$  given in (1.1). We omit the details.

To end this introduction, let us comment on previous results. The limit as  $p \rightarrow \infty$  of the first eigenvalue  $\lambda_p^D$  of the usual local  $p$ -Laplacian with Dirichlet boundary condition was studied in [27, 28], (see also [5] for an anisotropic version). In those papers the authors prove that

$$\lambda_\infty^D := \lim_{p \rightarrow +\infty} (\lambda_p^D)^{1/p} = \inf \left\{ \frac{\|\nabla v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} : v \in W_0^{1,\infty}(\Omega), v \not\equiv 0 \right\} = \frac{1}{R},$$

where  $R$  is the largest possible radius of a ball contained in  $\Omega$ . In addition, it was shown the existence of extremals, i.e. functions where the above infimum is attained. These extremals can be constructed taking the limit as  $p \rightarrow \infty$  in the eigenfunctions of the  $p$ -Laplacian eigenvalue problems (see [27]) and are viscosity solutions of the following eigenvalue problem (called the infinity eigenvalue problem in the literature)

$$\begin{cases} \min \{|Du| - \lambda_\infty^D u, \Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The limit operator  $\Delta_\infty$  that appears here is the  $\infty$ -Laplacian given by  $\Delta_\infty u = -\langle D^2 u Du, Du \rangle$ . Remark that solutions to  $\Delta_p v_p = 0$  with a Dirichlet data  $v_p = f$  on  $\partial\Omega$  converge as  $p \rightarrow \infty$  to the viscosity solution to  $\Delta_\infty v = 0$  with  $v = f$  on  $\partial\Omega$ , see [4, 6, 13]. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in  $\Omega$  of a boundary data  $f$ , see [2, 4]. Limits of  $p$ -Laplacians are also relevant in mass transfer problems, see [7, 19].

On the other hand, the pseudo infinity Laplacian is the second order nonlinear operator given by  $\tilde{\Delta}_\infty u = \sum_{i \in I(\nabla u)} u_{x_i x_i} |u_{x_i}|^2$ , where the sum is taken over the indexes in  $I(\nabla u) = \{i : |u_{x_i}| = \max_j |u_{x_j}|\}$ . This operator, as happens for the usual infinity Laplacian, also appears naturally as a limit of  $p$ -Laplace type problems. In fact, any possible limit of  $u_p$ , solutions to  $\tilde{\Delta}_p u = \sum_{i=1}^N (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0$ , is a viscosity solution to  $\tilde{\Delta}_\infty u = 0$ . A proof of this fact is contained in [5], where are also studied the eigenvalue problem for this operator.

Concerning regularity, we mention [35] where it is proved that infinity harmonic functions, that is, viscosity solutions to  $-\Delta_\infty u = 0$ , are  $C^1$  in two dimensions and [20, 21] where it is proved differentiability in any dimension. For the pseudo infinity Laplacian, we refer here to solutions to  $\tilde{\Delta}_\infty u = 0$ , the optimal regularity is Lipschitz continuity, see [34].

For references concerning nonlocal fractional problems we refer to [18, 26, 29, 30, 32, 31, 17] and references therein. For limits as  $p \rightarrow +\infty$  in nonlocal  $p$ -Laplacian problems and its relation with optimal mass transport we refer to [26] (eigenvalue problems were not considered there).

Finally, concerning limits as  $p \rightarrow \infty$  in fractional eigenvalue problems, we mention [9, 23, 28]. In [28] the limit of the first eigenvalue for the fractional  $p$ -Laplacian is studied while in [23] higher eigenvalues are considered. We borrow ideas and techniques from these papers. In particular, when we prove the fact that there is a limit problem that is verified in the viscosity sense. For example, the fact that continuous weak solutions to our pseudo fractional  $p$ -Laplacian are viscosity solutions runs exactly as in [28] and hence we omit the details here.

The paper is organized as follows: In Section 2 we collect some preliminary results; in Section 3 we deal with our eigenvalue problem and prove Theorem 1.1; in Section 4 we analyze the limit as  $s \rightarrow 1^-$ , Theorem 1.2; finally, in Section 5 we study the limit as  $p \rightarrow \infty$  proving Theorem 1.3.

## 2. PRELIMINARIES

Throughout this section  $s \in (0, 1)$ ,  $p \in (1, +\infty]$ ,  $\Omega$  is an open set of  $\mathbb{R}^{n+m}$ . We henceforth use the notation:

- $(x, y) = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \in \mathbb{R}^{n+m}$  with  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $y = (x_{n+1}, \dots, x_{n+m}) \in \mathbb{R}^m$ ;
- $\Omega^2 = \Omega \times \Omega$ ;
- $\Omega_x = \{y \in \mathbb{R}^m : (x, y) \in \Omega\}$ , and  $\Omega_y = \{x \in \mathbb{R}^n : (x, y) \in \Omega\}$ ;
- $B^N(x, r)$  denotes the ball of  $N$ -ball of radius  $r$  and center  $x$ , and  $\omega_N$  denotes the  $(N - 1)$ -dimensional Hausdorff measure of the  $N$ -sphere of radius 1;
- $(a)^{p-1} = |a|^{p-2}a$ .

Given a measurable function  $u: \Omega \rightarrow \mathbb{R}$ , we set for  $p < +\infty$ ,

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &:= \int_{\Omega} |u(x, y)|^p dx dy, \\ |u|_{W^{s,p}(\Omega)}^p &= \int_{\Omega^2} \frac{|u(x, y) - u(z, w)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dx dy dz dw, \\ [u]_{W^{s,p}(\Omega)}^p &= \int_{\Omega} \int_{\Omega_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \\ &\quad + \int_{\Omega} \int_{\Omega_x} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dw dx dy \end{aligned}$$

and for  $p = +\infty$ ,

$$\begin{aligned} |u|_{W^{s,\infty}(\Omega)} &= \sup \left\{ \frac{|u(x, y) - u(z, y)|}{|(x, y) - (z, y)|^s} : (x, y) \neq (z, y) \in \Omega \right\} = |u|_{C^{0,s}(\Omega)}, \\ [u]_{W^{s,\infty}(\Omega)} &= \max \left\{ \sup \left\{ \frac{|u(x, y) - u(z, y)|}{|x - z|^s} : (x, y) \neq (z, y) \in \Omega \right\}; \right. \\ &\quad \left. \sup \left\{ \frac{|u(x, y) - u(x, w)|}{|y - w|^s} : (x, y) \neq (x, w) \in \Omega \right\} \right\}. \end{aligned}$$

We denote by  $W^{s,p}(\Omega)$  (here  $p$  can be  $+\infty$ ) the usual fractional Sobolev space, that is  $W^{s,p}(\Omega) := \{u \in L^p(\Omega) : |u|_{W^{s,p}(\Omega)} < +\infty\}$ .

We introduce the space  $\mathcal{W}^{s,p}(\Omega)$  (again here  $p$  can be  $+\infty$ ) as follows:

$$\mathcal{W}^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : [u]_{\mathcal{W}^{s,p}(\Omega)}^p < \infty \right\}.$$

This space is a Banach space. We state this as a proposition but we omit its proof that is standard.

**Proposition 2.1.** *The space  $\mathcal{W}^{s,p}(\Omega)$  endowed with the norm*

$$\|u\|_{\mathcal{W}^{s,p}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + [u]_{\mathcal{W}^{s,p}(\Omega)}^p \right)^{1/p}$$

*is a Banach space. Moreover  $\mathcal{W}^{s,p}(\Omega)$  is separable for  $1 \leq p < +\infty$  and it is reflexive for  $1 < p < \infty$ .*

For  $u: \Omega \rightarrow \mathbb{R}$  a measurable function, we set

$$u_+(x, y) = \max\{u(x, y), 0\} \quad \text{and} \quad u_-(x, y) = \min\{-u(x, y), 0\}.$$

Observe that

$$|u_{\pm}(x, y) - u_{\pm}(z, w)| \leq |u(x, y) - u(z, w)|$$

for all  $(x, y), (z, w) \in \Omega$ . Therefore, we have

**Lemma 2.2.** *Let  $\mathcal{X} = W^{s,p}(\Omega)$  or  $\mathcal{W}^{s,p}(\Omega)$ . If  $u \in \mathcal{X}$  then  $u_+, u_- \in \mathcal{X}$ .*

For  $1 \leq p < \infty$ , we denote by  $\widetilde{\mathcal{W}}^{s,p}(\Omega)$  the space of all  $u \in \mathcal{W}^{s,p}(\Omega)$  such that  $\tilde{u} \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m})$  where  $\tilde{u}$  is the extension by zero of  $u$ .

The next result can be found in [1, 15].

**Theorem 2.3.** *Under the assumptions A1 and A2 we have that*

- *If  $sp < n + m$ , then  $W^{s,p}(\Omega)$  is compactly embedded in  $L^q(\Omega)$  for all  $1 \leq q < p_s^* = (n+m)p/(n+m-sp)$ .*
- *If  $sp = n + m$ , then  $W^{s,p}(\Omega)$  is compactly embedded in  $L^q(\Omega)$  for all  $1 \leq q < \infty$ .*
- *If  $sp > n + m$ , then  $W^{s,p}(\Omega)$  is compactly embedded in  $C^{0,\lambda}(\overline{\Omega})$  with  $\lambda < s - (n+m)/p$ .*

**Lemma 2.4.** *Let  $\Omega_1$  and  $\Omega_2$  be open subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively. If  $\Omega = \Omega_1 \times \Omega_2$ , and  $p \in [1, +\infty)$ , then  $\mathcal{W}^{s,p}(\Omega)$  is continuously embedded in  $W^{s,p}(\Omega)$ . Moreover, there exists a constant  $C = C(n, m)$  such that*

$$|u|_{W^{s,p}(\Omega)}^p \leq C[u]_{\mathcal{W}^{s,p}(\Omega)}^p$$

for all  $u \in \mathcal{W}^{s,p}(\Omega)$ .

*Proof.* Let  $u \in \mathcal{W}^{s,p}(\Omega)$ . We have

$$\begin{aligned} |u|_{W^{s,p}(\Omega)}^p &= \int_{\Omega^2} \frac{|u(x, y) - u(z, w)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dx dy dz dw \\ &\leq 2^{p-1} \int_{\Omega^2} \frac{|u(x, y) - u(z, y)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dx dy dz dw \\ &\quad + 2^{p-1} \int_{\Omega^2} \frac{|u(z, y) - u(z, w)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dx dy dz dw \\ &= 2^{p-1} I_1 + 2^{p-1} I_2. \end{aligned} \tag{2.1}$$

Now, we observe that

$$\begin{aligned}
I_1 &= \int_{\Omega^2} \frac{|u(x, y) - u(z, y)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dx dy dz dw \\
&\leq \int_{\Omega} \int_{\Omega_2} \int_{\mathbb{R}^m} \frac{|u(x, y) - u(z, y)|^p}{|(x, y) - (z, w)|^{n+m+sp}} dw dz dx dy \\
&\leq \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} \int_{\mathbb{R}^m} \frac{|x - z|^{n+sp} dw}{(|x - z|^2 + |y - w|^2)^{\frac{n+m+sp}{2}}} dz dx dy \\
&= \omega_m \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \int_0^{+\infty} \frac{r^{m-1}}{(1+r^2)^{\frac{n+m+sp}{2}}} dr.
\end{aligned}$$

Since

$$\int_0^{+\infty} \frac{r^{m-1}}{(1+r^2)^{\frac{n+m+sp}{2}}} dr \leq \int_0^1 r^{m-1} dr + \int_1^{+\infty} \frac{1}{r^{n+sp+1}} dr = \frac{1}{m} + \frac{1}{n+sp}$$

we have that

$$(2.2) \quad I_1 \leq 2\omega_m \int_{\Omega} \int_{\Omega_2} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy.$$

One can also, in an analogous way, obtain

$$(2.3) \quad I_2 \leq 2\omega_n \int_{\Omega} \int_{\Omega_1} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dw dx dy.$$

By (2.1), (2.2) and (2.3), we get

$$|u|_{W^{s,p}(\Omega)} \leq C(n, m)[u]_{\mathcal{W}^{s,p}(\Omega)}.$$

This completes the proof.  $\square$

*Remark 2.5.* If  $p = \infty$ , it is straightforward to show that  $W^{s,\infty}(\Omega) \subset \mathcal{W}^{s,\infty}(\Omega)$ . Moreover, if  $\Omega = \Omega_1 \times \Omega_2$  then  $\mathcal{W}^{s,\infty}(\Omega) = W^{s,\infty}(\Omega)$ .

**Lemma 2.6.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^{n+m}$  and  $p \in (1, \infty)$ . If  $0 < t < s < 1$  then  $\mathcal{W}^{s,p}(\Omega) \subset \mathcal{W}^{t,p}(\Omega)$ , and the embedding is continuous. Moreover*

$$(2.4) \quad [u]_{\mathcal{W}^{t,p}(\Omega)}^p \leq [u]_{\mathcal{W}^{s,p}(\Omega)}^p + \frac{2^p(\omega_n + \omega_m)}{tp} \|u\|_{L^p(\Omega)}^p \quad \forall u \in \mathcal{W}^{s,p}(\Omega).$$

*Proof.* Let  $u \in \mathcal{W}^{s,p}(\Omega)$ . Observe that,

$$\begin{aligned}
\int_{\Omega} \int_{\Omega_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dz dx dy &\leq \int_{\Omega} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dz dx dy \\
&\quad + \int_{\Omega} \int_{A_y^c} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dz dx dy
\end{aligned}$$

where  $A_y = \{z \in \Omega_y : |z - x| < 1\}$ . Since  $t < s$ , we have that

$$\begin{aligned}
& \int_{\Omega} \int_{\Omega_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+tp}} dz dx dy \leq \\
& \leq \int_{\Omega} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy + 2^{p-1} \int_{\Omega} \int_{A_y^c} \frac{|u(x, y)|^p + |u(z, y)|^p}{|x - z|^{n+tp}} dz dx dy \\
& \leq \int_{\Omega} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy + 2^p \int_{\Omega} \int_{A_y^c} \frac{|u(x, y)|^p}{|x - z|^{n+tp}} dz dx dy \\
& \leq \int_{\Omega} \int_{A_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy + \frac{2^p \omega_n}{tp} \int_{\Omega} |u(x, y)|^p dx dy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{\Omega} \int_{\Omega_x} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{n+tp}} dz dx dy \leq \\
& \leq \int_{\Omega} \int_{A_x} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy + \frac{2^p \omega_m}{tp} \int_{\Omega} |u(x, y)|^p dx dy,
\end{aligned}$$

where  $A_x = \{w \in \Omega_x : |y - w| < 1\}$ . Therefore (2.4) holds.  $\square$

Finally, we prove a Poincaré type inequality.

**Lemma 2.7.** *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^{n+m}$ ,  $s \in (0, 1)$  and  $p \in (1, \infty)$ . Then there is a positive constant  $C$  such that*

$$\|u\|_{L^p(\Omega)} \leq C[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \quad \forall u \in \widetilde{\mathcal{W}}^{s,p}(\Omega).$$

*Proof.* Let  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$  and  $d = 2 \operatorname{diam}(\Omega)$ . It holds that

$$[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \geq \int_{\Omega} |u(x, y)|^p \int_{\mathbb{R}^{n+m} \setminus B^n(x, d)} \frac{dz}{|x - z|^{n+sp}} \geq \frac{\omega_n d^{-sp}}{sp} \|u\|_{L^p(\Omega)}^p.$$

$\square$

### 3. THE FIRST EIGENVALUE

Under assumptions A1 and A2, a natural definition of an eigenvalue is a real value  $\lambda$  for which there exists  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$  such that  $u$  is a weak solution of

$$(3.1) \quad \begin{cases} \mathcal{L}_{s,p} u(x, y) = \lambda (u(x, y))^{p-1} & \text{in } \Omega, \\ u(x, y) = 0 & \text{in } \Omega^c, \end{cases}$$

that is

$$\mathcal{H}_{s,p}(u, v) = \lambda \int_{\Omega} (u(x, y))^{p-1} v(x, y) dx dy \quad \forall v \in \widetilde{\mathcal{W}}^{s,p}(\Omega).$$

The function  $u$  is called a corresponding eigenfunction. Here

$$\begin{aligned}
\mathcal{H}_{s,p}(u, v) &:= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{(u(x, y) - u(z, y))^{p-1} (v(x, y) - v(z, y))}{|x - z|^{n+sp}} dz dx dy \\
&+ \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{(u(x, y) - u(x, w))^{p-1} (v(x, y) - v(x, w))}{|y - w|^{m+sp}} dw dx dy.
\end{aligned}$$



Observe that

$$\mathcal{H}_{s,p}(u, u) = [u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \quad \forall u \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}),$$

and, by Hölder's inequality,

$$\mathcal{H}_{s,p}(u, v) \leq 2[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^{p-1} [v]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} \quad \forall u, v \in \mathcal{W}^{s,p}(\mathbb{R}^{n+m}).$$

Observe that, when  $\lambda$  is an eigenvalue, then there is  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\}$  such that

$$\mathcal{H}_{s,p}(u, u) = \lambda \int_{\Omega} |u(x, y)|^p dx dy.$$

Then, we have that

$$\lambda = \frac{[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} \geq 0.$$

By a standard compactness argument, we have the following result.

**Theorem 3.1.** *Under the assumptions A1 and A2, the first eigenvalue is given by*

$$\lambda_1(s, p) := \inf \left\{ \frac{[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} : u \in \widetilde{\mathcal{W}}^{s,p}(\Omega), u \neq 0 \right\}.$$

*Proof.* Consider a minimizing sequence  $u_n$  normalized according to  $\|u_n\|_{L^p(\Omega)} = 1$ . Then, as  $u_n$  is bounded in  $\widetilde{\mathcal{W}}^{s,p}(\Omega)$ , by Lemma 2.4 and Theorem 2.3, there is a subsequence such that  $u_{n_j} \rightharpoonup u$  weakly in  $\widetilde{\mathcal{W}}^{s,p}(\Omega)$  and  $u_{n_j} \rightarrow u$  strongly in  $L^p(\Omega)$ . Therefore,  $u$  is a nontrivial minimizer to the variational problem defining  $\lambda_1(s, p)$ . The fact that this minimizer is a weak solution to (3.1) is straightforward and can be obtained from the arguments in [29].

To finish the proof we just observe that any other eigenfunction associated with an eigenvalue  $\lambda$  verifies

$$\lambda = \frac{[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p} \geq \lambda_1(s, p),$$

and then we get that  $\lambda_1(s, p)$  is the first eigenvalue.  $\square$

Now we observe that using a topological tool (the genus) we can construct an unbounded sequence of eigenvalues.

**Theorem 3.2.** *Assume A1 and A2. There is a sequence of eigenvalues  $\lambda_n$  such that  $\lambda_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .*

*Proof.* We follow ideas from [22] and hence we omit the details. Let us consider

$$M_\alpha = \{u \in \widetilde{\mathcal{W}}^{s,p}(\Omega) : [u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})} = p\alpha\}$$

and

$$\varphi(u) = \frac{1}{p} \int_{\Omega} |u(x, y)|^p dx dy.$$

We are looking for critical points of  $\varphi$  restricted to the manifold  $M_\alpha$  using a minimax technique. We consider the class

$$\Sigma = \{A \subset \widetilde{\mathcal{W}}^{s,p}(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A\}.$$

Over this class we define the genus,  $\gamma : \Sigma \rightarrow \mathbb{N} \cup \{\infty\}$ , as

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{there exists } \phi \in C(A, \mathbb{R}^k - \{0\}), \phi(x) = -\phi(-x)\}.$$

Now, we let  $C_k = \{C \subset M_\alpha : C \text{ is compact, symmetric and } \gamma(C) \leq k\}$  and let

$$\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u).$$

Then  $\beta_k > 0$  and there exists  $u_k \in M_\alpha$  such that  $\varphi(u_k) = \beta_k$  and  $u_k$  is a weak eigenfunction with  $\lambda_k = \alpha/\beta_k$ .  $\square$

The following lemma shows that the eigenfunctions are bounded.

**Lemma 3.3.** *Under assumptions A1 and A2, if  $u$  is an eigenfunction associated to some eigenvalue  $\lambda$  then  $u \in L^\infty(\mathbb{R}^{n+m})$ .*

*Proof.* In this proof we follow ideas from [23].

If  $ps > n + m$ , by Lemma 2.4 and Theorem 2.3, then the assertion holds. From now on, we suppose that  $sp \leq n + m$ .

We will show that if  $\|u_+\|_{L^p(\Omega)} \leq \delta$  then  $u_+$  is bounded, where  $\delta > 0$  is some small constant to be determined. Let  $k \in \mathbb{N}_0$ , we define the function  $u_k$  by

$$u_k(x, y) := (u(x, y) - 1 + 2^{-k})_+.$$

Observe that,  $u_0 = u_+$  and for any  $k \in \mathbb{N}_0$  we have that  $u_k \in \widetilde{W}^{s,p}(\Omega)$  verifies

$$\begin{aligned} (3.2) \quad & u_{k+1} \leq u_k \text{ a.e. } \mathbb{R}^{n+m}, \\ & u < (2^{k+1} - 1)u_k \text{ in } \{u_{k+1} > 0\}, \\ & \{u_{k+1} > 0\} \subset \{u_k > 2^{-(k+1)}\}. \end{aligned}$$

Now, for any function  $v : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ , it holds that

$$|v_+(x, y) - v_+(z, w)|^p \leq |v(x, y) - v(z, w)|^{p-1} (v_+(x, y) - v_+(z, w))$$

for all  $(x, y), (z, w) \in \mathbb{R}^{n+m}$ . Then

$$[u_{k+1}]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \leq \mathcal{H}_{s,p}(u, u_{k+1}) = \lambda \int_{\Omega} (u(x, y))^{p-1} u_{k+1}(x, y) dx dy$$

for all  $k \in \mathbb{N}_0$ . Hence, by (3.2) and Hölder's inequality, we get

$$\begin{aligned} (3.3) \quad & [u_{k+1}]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \leq \lambda \int_{\Omega} (u(x, y))^{p-1} u_{k+1}(x, y) dx dy \\ & \leq (2^{k+1} - 1)^{p-1} \lambda \|u_k\|_{L^p(\Omega)}^p \end{aligned}$$

for all  $k \in \mathbb{N}_0$ .

On the other hand, in the case  $sp < n + m$ , using Hölder's inequality, Lemma 2.4 and Theorem 2.3, the formulas in (3.2), and Chebyshev's inequality, for any  $k \in \mathbb{N}_0$

we have that

$$\begin{aligned}
 \|u_{k+1}\|_{L^p(\Omega)}^p &\leq \|u_{k+1}\|_{L^{p^*}(\Omega)}^p |\{u_{k+1} > 0\}|^{sp/(n+m)} \\
 (3.4) \quad &\leq C[u_{k+1}]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p |\{u_k > 2^{-(k+1)}\}|^{sp/(n+m)} \\
 &\leq C[u_{k+1}]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \left(2^{(k+1)p} \|u_k\|_{L^p(\Omega)}^p\right)^{sp/(n+m)},
 \end{aligned}$$

where  $C$  is a constant independent of  $k$ . Then, by (3.3) and (3.4), for any  $k \in \mathbb{N}_0$  we obtain

$$(3.5) \quad \|u_{k+1}\|_{L^p(\Omega)}^p \leq C \left(2^{(k+1)p} \|u_k\|_{L^p(\Omega)}^p\right)^{1+\alpha},$$

where  $C$  is a constant independent of  $k$  and  $\alpha = sp/(n+m) > 0$ .

Arguing similarly, in the case  $sp = n + m$ , taking  $r > p$  and proceeding as in the previous case,  $sp < n + m$  (with  $r$  in place of  $p_s^*$ ), we obtain that (3.5) holds with  $\alpha = 1 - p/r > 0$ .

Therefore, if  $sp \leq n + m$ , there exist  $\alpha > 0$  and a constant  $C > 1$  such that

$$\|u_{k+1}\|_{L^p(\Omega)}^p \leq C^k \left(\|u_k\|_{L^p(\Omega)}^p\right)^{1+\alpha},$$

for any  $k \in \mathbb{N}_0$ . Hence, if  $\|u_0\|_{L^p(\Omega)}^p = \|u_+\|_{L^p(\Omega)}^p \leq C^{-1/\alpha^2} =: \delta^p$  then  $u_k \rightarrow 0$  strongly in  $L^p(\Omega)$ . But  $u_k \rightarrow (u-1)_+$  a.e in  $\mathbb{R}^{n+m}$ , then we conclude that  $(u-1)_+ \equiv 0$  in  $\mathbb{R}^{n+m}$ . Therefore,  $u_+$  is bounded.

Taking  $-u$  in place of  $u$  we have that  $u_-$  is bounded if  $\|u_-\|_{L^p(\Omega)} < \delta$ .

Hence, as we can multiply an eigenfunction  $u$  by a small constant in order to obtain  $\|u_+\|_{L^p(\Omega)}$  and  $\|u_-\|_{L^p(\Omega)} < \delta$ , we conclude that  $u$  is bounded.  $\square$

Our next goal is to show that if  $u$  is a eigenfunction associated with  $\lambda_1(s, p)$  then  $u$  does not change sign. Before showing this result we need the following two technical lemmas.

**Lemma 3.4.** *Assume A1 and A2. If  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$  is such that*

$$(3.6) \quad \mathcal{H}_{s,p}(u, v) \geq 0 \quad \forall v \in \widetilde{\mathcal{W}}^{s,p}(\Omega), v \geq 0 \text{ in } \Omega.$$

*and  $u \geq 0$  in  $B^n(x_0, R) \times B^m(y_0, R) \subset \subset \Omega$  for some  $R > 0$  then for any  $d > 0$  and  $0 < 2r < R$  there holds*

$$\begin{aligned}
 &\int_{B_r^n} \int_{B_r^n} \int_{B_r^n} \frac{1}{|x-z|^{n+sp}} \left| \log \left( \frac{u(x,y)+d}{u(z,y)+d} \right) \right|^p dz dx dy \\
 &+ \int_{B_r^n} \int_{B_r^n} \int_{B_r^n} \frac{1}{|y-w|^{m+sp}} \left| \log \left( \frac{u(x,y)+d}{u(x,w)+d} \right) \right|^p dw dx dy \\
 (3.7) \quad &\leq C r^{n+m-sp} \left\{ \frac{r^{sp}}{d^{p-1} r^m} \int_{\mathbb{R}^m} \int_{(B_R^n)^c} \frac{u_-(x,y)^p}{|x-x_0|^{n+sp}} dx dy \right. \\
 &\quad \left. + \frac{r^{sp}}{d^{p-1} r^n} \int_{\mathbb{R}^n} \int_{(B_R^m)^c} \frac{u_-(x,y)^p}{|y-y_0|^{m+sp}} dy dx + 1 \right\}
 \end{aligned}$$

where  $B_\rho^n = B^n(x_0, \rho)$ ,  $B_\rho^m = B^m(y_0, \rho)$  and  $C = C(n, m, p, s) > 0$  is a constant.

*Proof.* Let  $d > 0$ ,  $r \in (0, R/2)$ ,

$$\begin{aligned} \phi &\in C_0^\infty(B_{3r/2}^n), \quad 0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_r^n, \quad |D_x \phi| < \frac{c}{r} \text{ in } B_{3r/2}^n, \text{ and} \\ \psi &\in C_0^\infty(B_{3r/2}^m), \quad 0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_r^m, \quad |D_x \psi| < \frac{c}{r} \text{ in } B_{3r/2}^m. \end{aligned}$$

Taking  $v(x, y) = \phi^p(x) \psi^p(y) (u(x, y) + d)^{1-p}$  as test function in (3.6) and following the proof of Lemma 1.3 in [16], we get (3.7).  $\square$

**Lemma 3.5.** *Assume A1 and A2. If  $\Omega$  is connected and  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$  is such that*

$$\mathcal{H}_{s,p}(u, v) \geq 0 \quad \forall v \in \widetilde{\mathcal{W}}^{s,p}(\Omega), v \geq 0 \text{ in } \Omega,$$

*$u \geq 0$  in  $\Omega$  and  $u \not\equiv 0$  in  $\Omega$  then  $u > 0$  in  $\Omega$ .*

*Proof.* In this proof we borrow ideas from [8]. Since  $\Omega$  is a bounded connected open set, it is enough to prove that  $u > 0$  in  $K$  for any  $K \subset\subset \Omega$  a connected compact set such that  $u \not\equiv 0$  in  $K$ .

Let  $K \subset\subset \Omega$  be a connected compact set such that  $u \not\equiv 0$  in  $K$ . Then there exists  $r > 0$  such that

$$K \subset \left\{ (x, y) \in \Omega : \max_{(z,w) \in \partial\Omega} \{|z - x|, |w - y|\} > 2r \right\}.$$

Since  $K$  is compact, there exists  $\{(x_j, y_j)\}_{j=1}^k \subset K$  such that

$$(3.8) \quad K \subset \bigcup_{j=1}^k B_j^n \times B_j^m, \quad \text{and} \quad |(B_j^n \times B_j^m) \cap (B_{j+1}^n \times B_{j+1}^m)| > 0$$

for any  $j \in \{1, \dots, k-1\}$ , where  $B_j^n = B^n(x_j, r/2)$  and  $B_j^m = B^m(y_j, r/2)$ .

To obtain a contradiction, suppose that  $|\{(x, y) : u(x, y) = 0\} \cap K| > 0$  then there exists  $j \in \{1, \dots, k\}$  such that

$$Z = \{(x, y) : u(x, y) = 0\} \cap (B_j^n \times B_j^m)$$

has positive measure.

Given  $d > 0$ , we define

$$F_d : B_j^n \times B_j^m \rightarrow \mathbb{R} \quad \text{by} \quad F_d(x, y) = \log \left( 1 + \frac{u(x, y)}{d} \right).$$

Then, for any  $(x, y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)$  and  $(z, w) \in Z$  we have

$$\begin{aligned}
F_d(z, w) &= 0 \\
|F_d(x, y)|^p &= |F(x, y) - F(z, w)|^p \\
&\leq 2^{p-1} \frac{|F(x, y) - F(z, y)|^p}{|z - x|^{n+sp}} |z - x|^{n+sp} \\
&\quad + 2^{p-1} \frac{|F(z, y) - F(z, w)|^p}{|w - y|^{m+sp}} |w - y|^{m+sp} \\
&\leq 2^{p-1} r^{n+sp} \frac{|F(x, y) - F(z, y)|^p}{|z - x|^{n+sp}} \\
&\quad + 2^{p-1} r^{m+sp} \frac{|F(z, y) - F(z, w)|^p}{|w - y|^{m+sp}} \\
&= 2^{p-1} r^{n+sp} \left| \log \left( \frac{u(x, y) + d}{u(z, y) + d} \right) \right|^p \frac{1}{|z - x|^{n+sp}} \\
&\quad + 2^{p-1} r^{m+sp} \left| \log \left( \frac{u(z, y) + d}{u(z, w) + d} \right) \right|^p \frac{1}{|w - y|^{m+sp}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|Z| |F_d(x, y)|^p &= \iint_Z |F_d(x, y)|^p dw dz \\
&\leq c_1 r^{n+m+sp} \int_{B_j^n} \left| \log \left( \frac{u(x, y) + d}{u(z, y) + d} \right) \right|^p \frac{dz}{|z - x|^{n+sp}} \\
&\quad + 2^{p-1} r^{m+sp} \int_{B_j^n} \int_{B_j^m} \left| \log \left( \frac{u(z, y) + d}{u(z, w) + d} \right) \right|^p \frac{dw dz}{|w - y|^{m+sp}}
\end{aligned}$$

for any  $(x, y) \in B^n(x_j, r/2) \times B^m(y_j, r/2)$ . Here  $c_1 = c_1(m, p) > 0$  is a constant.

Then

$$\begin{aligned}
&\int_{B_j^n} \int_{B_j^m} |F_d(x, y)|^p dx dy \\
&\leq \frac{c_1 r^{n+m+sp}}{|Z|} \int_{B_j^m} \int_{B_j^n} \int_{B_j^n} \left| \log \left( \frac{u(x, y) + d}{u(z, y) + d} \right) \right|^p \frac{dz dx dy}{|z - x|^{n+sp}} \\
&\quad + \frac{c_2 r^{n+m+sp}}{|Z|} \int_{B_j^n} \int_{B_j^m} \int_{B_j^m} \left| \log \left( \frac{u(x, y) + d}{u(x, w) + d} \right) \right|^p \frac{dw dx dy}{|w - y|^{m+sp}}.
\end{aligned}$$

Thus, by Lemma 3.4 and since  $u \geq 0$  in  $\Omega$ , we get

$$\int_{B_j^n} \int_{B_j^m} |F_d(x, y)|^p dx dy \leq C \frac{r^{2n+2m}}{|Z|},$$

where  $C = C(n, m, s, p) > 0$  is a constant. Taking  $d \rightarrow 0$  in the last inequality, we get that  $u \equiv 0$  in  $B_j^n \times B_j^m$ .

By (3.8), there exists  $i \in \{1, \dots, k\}$  such that  $i \neq j$  and

$$|(B_i^n \times B_i^m) \cap \{(x, y) : u(x, y) = 0\}| > 0.$$

Then, we can repeat the previous argument for  $B_i^n \times B_i^m$  and obtain  $u \equiv 0$  in  $B_i^n \times B_i^m$ . In this way we conclude that  $u \equiv 0$  in  $K$  which contradicts the fact that  $u \not\equiv 0$  in  $K$ . Thus  $|\{(x, y) : u(x, y) = 0\} \cap K| = 0$ .  $\square$

Now, we are ready to prove that the eigenfunctions associated to  $\lambda_1(s, p)$  do not change sign.

**Theorem 3.6.** *Assume A1 and A2. If  $u$  is an eigenfunction associated to  $\lambda_1(s, p)$  then  $|u| > 0$  in  $\Omega$ .*

*Proof.* We start by showing that if  $u$  is an eigenfunction corresponding to  $\lambda_1(s, p)$  then  $|u| \not\equiv 0$  in all connected components of  $\Omega$ . Our proof is by contradiction. We therefore assume that there is a connected component  $A$  of  $\Omega$  such that  $|u| \equiv 0$ . Since  $u$  is an eigenfunction corresponding to  $\lambda_1(s, p)$  then so is  $|u|$ . Then

$$\begin{aligned} 0 &= \lambda_1(s, p) \int_{\Omega} |u(x, y)|^{p-1} \phi(x, y) dx dy = \mathcal{H}_{s,p}(|u|, \phi) \\ &= -2 \int_{A^c} \int_{A_y} \frac{|u(x, y)|^{p-1} \phi(z, y)}{|x - z|^{n+sp}} dz dx dy - 2 \int_{A^c} \int_{A_x} \frac{|u(x, y)|^{p-1} \phi(x, w)}{|y - w|^{m+sp}} dw dx dy \end{aligned}$$

for all  $\phi \in C_0^\infty(A)$ , which is a contradiction.

Therefore, if  $A$  connected components  $C$  of  $\Omega$  then  $|u| \not\equiv 0$  in  $A$  and

$$\mathcal{H}_{s,p}(|u|, v) = \lambda_1(s, p) \int_{\Omega} |u(x, y)|^{p-1} v(x, y) dx dy \geq 0 \quad \forall v \in \widetilde{\mathcal{W}}^{s,p}(A).$$

Then, by Lemma 3.5,  $|u| > 0$  in  $A$ . Therefore  $|u| > 0$  in  $\Omega$ .  $\square$

Our next result show that  $\lambda_1(s, p)$  is simple.

**Theorem 3.7.** *Assume A1 and A2. Let  $u$  be a positive eigenfunction corresponding to  $\lambda_1(s, p)$ . If  $\lambda > 0$  is such that there exists a non-negative eigenfunction  $v$  of (3.1) with eigenvalue  $\lambda$ , then  $\lambda = \lambda_1(s, p)$  and there exists  $k \in \mathbb{R}$  such that  $v = ku$  a.e. in  $\Omega$ .*

*Proof.* Since  $\lambda_1(s, p)$  is the first eigenvalue we have that  $\lambda_1(s, p) \leq \lambda$ . Let  $k \in \mathbb{N}$  and define  $v_k := v + 1/k$ .

We begin proving that  $w_k := u^p / v_k^{p-1} \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ . It is immediate that  $w_k = 0$  in  $\Omega^c$  and  $w_k \in L^p(\Omega)$ , due to the fact that  $u \in L^\infty(\Omega)$ , see Lemma 3.3.

On the other hand

$$\begin{aligned} &|w_k(x, y) - w_k(z, w)| \\ &= \left| \frac{u(x, y)^p - u(z, w)^p}{v_k(x, y)^{p-1}} + \frac{u(z, w)^p (v_k(z, w)^{p-1} - v_k(x, y)^{p-1})}{v_k(x, y)^{p-1} v_k(z, w)^{p-1}} \right| \\ &\leq k^{p-1} |u(x, y)^p - u(z, w)^p| + \|u\|_{L^\infty(\Omega)}^p \frac{|v_k(x, y)^{p-1} - v_k(z, w)^{p-1}|}{v_k(x, y)^{p-1} v_k(z, w)^{p-1}} \\ &\leq 2 \|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} p |u(x, y) - u(z, w)| \\ &\quad + \|u\|_{L^\infty(\Omega)}^p (p-1) \frac{v_k(x, y)^{p-2} + v_k(z, w)^{p-2}}{v_k(x, y)^{p-1} v_k(z, w)^{p-1}} |v_k(x, y) - v_k(z, w)| \\ &\leq 2 \|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} p |u(x, y) - u(z, w)| \\ &\quad + \|u\|_{L^\infty(\Omega)}^p (p-1) k^{p-1} \left( \frac{1}{v_k(x, y)} + \frac{1}{v_k(z, w)} \right) |v(y) - v(x)| \\ &\leq C(k, p, \|u\|_{L^\infty(\Omega)}) (|u(x, y) - u(z, w)| + |v(x, y) - v(z, w)|) \end{aligned}$$

for all  $(x, y), (z, w) \in \mathbb{R}^{n+m}$ . Hence, we have that  $w_k \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$  for all  $k \in \mathbb{N}$  since  $u, v \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ .

Set

$$L(u, v_k)(x, y, z, w) = |u(x, y) - u(w, z)|^p - (v_k(x, y) - v_k(w, z))^{p-1} \left( \frac{u(x, y)^p}{v_k(x, y)^{p-1}} - \frac{u(z, w)^p}{v_k(z, w)^{p-1}} \right).$$

Then, by [2, Lemma 6.2] and since  $u, v$  are two positive eigenfunctions of problem (3.1) with eigenvalues  $\lambda_1(s, p)$  and  $\lambda$  respectively, we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{L(u, v_k)(x, y, z, y)}{|x - z|^{n+sp}} dz dxdy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{L(u, v_k)(x, y, x, w)}{|y - w|^{m+sp}} dw dxdy \\ &\leq \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dxdy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dw dxdy \\ &\quad - \mathcal{H}_{s,p}(v, w_k) \\ &\leq \lambda_1(s, p) \int_{\Omega} u(x, y)^p dxdy - \lambda \int_{\Omega} v(x, y)^{p-1} w_k(x, y) dxdy \\ &= \lambda_1(s, p) \int_{\Omega} u(x, y)^p dxdy - \lambda \int_{\Omega} v(x, y)^{p-1} \frac{u(x, y)^p}{v_k(x, y)^{p-1}} dxdy. \end{aligned}$$

By Fatou's lemma and the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{L(u, v)(x, y, z, y)}{|x - z|^{n+sp}} dz dxdy + \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{L(u, v)(x, y, x, w)}{|y - w|^{m+sp}} dw dxdy = 0$$

due to  $\lambda_1(s, p, h) \leq \lambda$ . Then  $L(u, v)(x, y, z, y) = L(u, v)(x, y, x, w) = 0$  a.e. Hence, again by Lemma 6.2 in [2],  $u(x, y) = \ell_1(y)v(x, y)$  and  $u(x, y) = \ell_2(x)v(x, y)$  for all  $(x, y) \in \mathbb{R}^{n+m}$ . Then, we conclude that  $u = \ell v$  for some constant  $\ell > 0$ .  $\square$

Finally we will prove that  $\lambda_1(s, p)$  is isolated.

**Theorem 3.8.** *Assume A1 and A2. Then  $\lambda_1(s, p)$  is isolated.*

*Proof.* We split the proof into two steps.

*Step 1.* If  $u$  is an eigenfunction associated to some eigenvalue  $\lambda > \lambda_1(s, p)$  then there is a positive constant  $C$  such that

$$(3.9) \quad \left( \frac{1}{C\lambda} \right)^{r/(r-p)} \leq |\Omega_{\pm}|$$

for all  $p < r < p_s^*$ . Here  $\Omega_{\pm} = \{(x, y) : u_{\pm} \neq 0\}$ , and

$$p_s^* = \begin{cases} \frac{(n+m)p}{n+m-sp}, & \text{if } sp < n+m, \\ \infty & \text{if } sp \geq n+m. \end{cases}$$

Let  $r \in (p, p_s^*)$ . By Theorem 2.3, Lemmas 2.7 and 2.4 and Hölder inequality, we have

$$\|u_+\|_{L^r(\Omega)}^p \leq C \|u_+\|_{W^{s,p}(\Omega)}^p \leq C \mathcal{H}_{s,p}(u, u_+) = C\lambda \|u_+\|_{L^r(\Omega)}^p |\Omega_+|^{(r-p)/r}.$$

Then

$$\left( \frac{1}{C\lambda} \right)^{r/(r-p)} \leq |\Omega_+|.$$

In order to prove the inequality for  $|\Omega_-|$ , it suffices to proceed as above, using the function  $-u$  instead of  $u$ .

*Step 2.* By definition,  $\lambda_1(s, p)$  is left-isolated. To prove that  $\lambda_1(s, p)$  is right-isolated, we argue by contradiction. We assume that there is a sequence of eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}}$  such that  $\lambda_k \searrow \lambda_1(s, p)$  as  $k \rightarrow \infty$ . Let  $u_k$  be an eigenfunction associated to  $\lambda_k$  such that  $\|u_k\|_{L^p(\Omega)} = 1$ . Then  $\{u_k\}_{k \in \mathbb{N}}$  is bounded in  $\widetilde{\mathcal{W}}^{s,p}(\Omega)$  and therefore we can extract a subsequence (that we still denoted by  $\{u_k\}_{k \in \mathbb{N}}$ ) such that

$$u_k \rightharpoonup u \text{ weakly in } \widetilde{\mathcal{W}}^{s,p}(\Omega), \quad u_k \rightarrow u \text{ strongly in } L^p(\Omega).$$

Then  $\|u\|_{L^p(\Omega)} = 1$  and

$$[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \leq \liminf_{k \rightarrow \infty} [u_k]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p = \lim_{k \rightarrow \infty} \lambda_k = \lambda_1(s, p).$$

Then  $u$  is an eigenfunction associated to  $\lambda_1(s, p)$ . Therefore  $u$  has constant sign.

Now, proceeding as in the proof of [3, Theorem 2], we arrive to a contradiction. In fact, by Egoroff's theorem we can find a subset  $A_\delta$  of  $\Omega$  such that  $|A_\delta| < \delta$  and  $u_k \rightarrow u$  uniformly in  $\Omega \setminus A_\delta$ . From (3.9) we get that  $u$  and the uniform convergence in  $\Omega \setminus A_\delta$  we obtain that  $|\{u > 0\}| > 0$  and  $|\{u > 0\}| < 0$ . This contradicts the fact that an eigenfunction associated with the first eigenvalue does not change sign.  $\square$

#### 4. THE LIMIT AS $s \rightarrow 1^-$

In this section, our goal is to show that

$$(4.1) \quad \begin{aligned} & \lim_{s \rightarrow 1^-} (1-s)\lambda_1(s, p) = \lambda_1(1, p) \\ & = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \left\{ \frac{K_{n,p} \int_{\Omega} |\nabla_x u(x, y)|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y u(x, y)|^p dx dy}{\|u\|_{L^p(\Omega)}^p} \right\} \end{aligned}$$

where  $K_{n,p}$  is a constant that depends only on  $n$  and  $p$ , and  $K_{m,p}$  depends only on  $m$  and  $p$ . Before proving (4.1), we need some technical results.

**Lemma 4.1.** *Let  $\Omega$  be an open subsets of  $\mathbb{R}^{n+m}$  with smooth boundary and  $p \in (1, \infty)$ . For all  $s \in (0, 1)$  we have that  $W^{1,p}(\Omega)$  is continuity embedded in  $\mathcal{W}^{s,p}(\Omega)$ .*

*Proof.* In this proof, we follow the ideas of the proof of [11, Theorem 1]. Let  $u \in W^{1,p}(\Omega)$ . By an extension argument, we can assume that  $u \in W^{1,p}(\mathbb{R}^{n+m})$ . We have that

$$(4.2) \quad \begin{aligned} & \int_{\mathbb{R}^{n+m}} |u(x+h, y) - u(x, y)|^p dx dy \leq |h|^p \int_{\mathbb{R}^{n+m}} |\nabla_x u(x, y)|^p dx dy, \\ & \int_{\mathbb{R}^{n+m}} |u(x, y+h) - u(x, y)|^p dx dy \leq |h|^p \int_{\mathbb{R}^{n+m}} |\nabla_y u(x, y)|^p dx dy. \end{aligned}$$

The proof of this fact can be carried out as that of Proposition XI.3 in [10] and is omitted.



Then, by (4.2), we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+m}} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dx dy dz \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+m}} \frac{|u(x + h, y) - u(x, y)|^p}{|h|^{n+sp}} dx dy dh \\
&\leq \int_{\{|h| \leq 1\}} \frac{dh}{|h|^{(s-1)p+n}} \int_{\mathbb{R}^{n+m}} |\nabla_x u(x, y)|^p dx dy \\
&\quad + 2 \int_{\{|h| > 1\}} \frac{dh}{|h|^{sp+n}} \int_{\mathbb{R}^{n+m}} |u(x, y)|^p dx dy \\
&\leq \frac{\omega_n}{(1-s)p} \int_{\mathbb{R}^{n+m}} |\nabla_x u(x, y)|^p dx dy + \frac{2\omega_n}{sp} \int_{\mathbb{R}^{n+m}} |u(x, y)|^p dx dy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n+m}} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dx dy dw \\
&\leq \frac{\omega_m}{(1-s)p} \int_{\mathbb{R}^{n+m}} |\nabla_y u(x, y)|^p dx dy + \frac{2\omega_m}{sp} \int_{\mathbb{R}^{n+m}} |u(x, y)|^p dx dy,
\end{aligned}$$

which completes the proof.  $\square$

*Remark 4.2.* Proceeding as in the proof of previous lemma and using the Poincaré inequality, we have that

$$(1-s)[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \leq C \left(1 + \frac{1}{s}\right) \int_{\Omega} |\nabla u|^p dx dy \quad \forall u \in W_0^{1,p}(\Omega)$$

where  $C$  is a constant independent of  $s$ .

**Lemma 4.3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^{n+m}$  with smooth boundary and  $p \in (1, \infty)$ . If  $u \in W_0^{1,p}(\Omega)$  then*

$$(1-s)[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \rightarrow K_{n,p} \int_{\Omega} |\nabla_x u|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y u|^p dx dy$$

as  $s \rightarrow 1^-$ .

*Proof.* We split the proof into two cases.

*Case 1.* First we prove the lemma for  $\phi \in C_0^\infty(\Omega)$ . Let  $B_1$  and  $B_2$  be two open balls in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively such that  $\Omega \subset B_1 \times B_2$ .

Given  $y \in B_2$ , we have that

$$\begin{aligned}
(4.3) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dx dz &= \int_{B_1} \int_{B_1} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dx dz \\
&\quad + 2 \int_{B_1} \int_{B_1^c} \frac{|\phi(x, y)|^p}{|x - z|^{n+sp}} dx dz.
\end{aligned}$$

By [11, Theorem 1], there is a constant  $K_{n,p}$  (that depends only the  $n$  and  $p$ ) such that

$$(4.4) \quad (1-s) \int_{B_1} \int_{B_1} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dx dz \rightarrow K_{n,p} \int_{B_1} |\nabla_x \phi(x, y)|^p dx$$

as  $s \rightarrow 1^-$ . On the other hand, since  $\text{supp}(\varphi) \subset\subset \Omega \subset B_1 \times B_2$ , there exists  $\delta > 0$  such that  $|x - z| > \delta$  for all  $z \in B_1^c$  and  $x \in \{t \in B_1 : (t, y) \in \text{supp}(\varphi)\}$ . Thus

$$(4.5) \quad (1-s) \int_{B_1} \int_{B_1^c} \frac{|\phi(x, y)|^p}{|x - z|^{n+sp}} dx dz \leq (1-s) \frac{\omega_n}{s p \delta^{sp}} \|\phi(\cdot, y)\|_{L^p(B_1)}^p \rightarrow 0$$

as  $s \rightarrow 1^-$ . Then by (4.3), (4.4), and (4.5) we have that

$$(4.6) \quad (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dx dz \rightarrow K_{n,p} \int_{B_1} |\nabla_x \phi(x, y)|^p dx$$

as  $s \rightarrow 1^-$ . Proceeding as in the proof of Lemma 4.1, we have that

$$\begin{aligned} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dx dz &\leq \frac{\omega_n}{p} \int_{\mathbb{R}^n} |\nabla_x \phi(x, y)|^p dx dy \\ &\quad + (1-s) \frac{2\omega_n}{s_0 p} \int_{\mathbb{R}^n} |\phi(x, y)|^p dx dy. \end{aligned}$$

Thus, (4.6) and the dominated convergence theorem imply

$$(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \rightarrow K_{n,p} \int_{\mathbb{R}^m} \int_{B_1} |\nabla_x \phi(x, y)|^p dx dy,$$

as  $s \rightarrow 1^-$ , that is,

$$(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|\phi(x, y) - \phi(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \rightarrow K_{n,p} \int_{\Omega} |\nabla_x \phi(x, y)|^p dx dy,$$

as  $s \rightarrow 1^-$ .

In the same manner we can see that there exists a constant  $K_{m,p}$  (that depends only the  $m$  and  $p$ ) such that

$$(1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^m} \frac{|\phi(x, y) - \phi(x, w)|^p}{|y - w|^{m+sp}} dw dx dy \rightarrow K_{m,p} \int_{\Omega} |\nabla_y \phi(x, y)|^p dx dy,$$

as  $s \rightarrow 1^-$ .

Then, we have

$$(1-s) [\phi]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p \rightarrow K_{n,p} \int_{\Omega} |\nabla_x \phi|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y \phi|^p dx dy,$$

as  $s \rightarrow 1^-$ .

*Case 2.* Now we prove the general case. Given  $u \in W_0^{1,p}(\Omega)$ , we define

$$\begin{aligned} F_s^u(x, y, z) &= (1-s)^{1/p} \frac{|u(x, y) - u(z, y)|}{|x - z|^{n/p+s}}, \\ G_s^u(x, y, z) &= (1-s)^{1/p} \frac{|u(x, y) - u(x, w)|}{|y - w|^{m/p+s}} \end{aligned}$$

and we want to show that

$$\|F_s^u\|_{L^p(\mathbb{R}^{2n+m})} \rightarrow K_{n,p}^{1/p} \|\nabla_x u\|_{L^p(\Omega)}, \quad \|G_s^u\|_{L^p(\mathbb{R}^{n+2m})} \rightarrow K_{m,p}^{1/p} \|\nabla_y u\|_{L^p(\Omega)},$$

as  $s \rightarrow 1^-$ .

Given  $\varepsilon > 0$  there is  $\phi \in C_0^\infty(\Omega)$  such that

$$\|\nabla u - \nabla \phi\|_{L^p(\Omega)} < \varepsilon.$$

Thus

$$(4.7) \quad ||\nabla_x u||_{L^p(\Omega)} - ||\nabla_x \phi||_{L^p(\Omega)}| < \varepsilon \text{ and } ||\nabla_x u||_{L^p(\Omega)} - ||\nabla_x \phi||_{L^p(\Omega)}| < \varepsilon.$$

By case 1, there exists  $s_0 \in (0, 1)$  such that

$$(4.8) \quad \begin{aligned} & ||F_s^\phi||_{L^p(\mathbb{R}^{2n+m})} - K_{n,p}^{1/p} ||\nabla_x \phi||_{L^p(\Omega)}| < \varepsilon, \\ & ||G_s^\phi||_{L^p(\mathbb{R}^{n+2m})} - K_{m,p}^{1/p} ||\nabla_y \phi||_{L^p(\Omega)}| < \varepsilon, \end{aligned}$$

for all  $s \in (s_0, 1)$ .

On the other hand, using Remark 4.2, we have that

$$(4.9) \quad \begin{aligned} & ||F_s^u||_{L^p(\mathbb{R}^{2n+m})} - ||F_s^\phi||_{L^p(\mathbb{R}^{2n+m})}| \leq C ||\nabla u - \nabla \phi||_{L^p(\Omega)} < C\varepsilon, \\ & ||G_s^u||_{L^p(\mathbb{R}^{2n+m})} - ||G_s^\phi||_{L^p(\mathbb{R}^{2n+m})}| \leq C ||\nabla u - \nabla \phi||_{L^p(\Omega)} < C\varepsilon, \end{aligned}$$

where  $C$  is a constant independent of  $s$ .

Finally, by (4.7), (4.8), and (4.9), we obtain that

$$\begin{aligned} & ||F_s^u||_{L^p(\mathbb{R}^{2n+m})} - K_{n,p}^{1/p} ||\nabla_x u||_{L^p(\Omega)}| < C\varepsilon, \\ & ||G_s^u||_{L^p(\mathbb{R}^{n+2m})} - K_{m,p}^{1/p} ||\nabla_y u||_{L^p(\Omega)}| < C\varepsilon, \end{aligned}$$

and the proof is complete.  $\square$

**Corollary 4.4.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^{n+m}$  with smooth boundary and  $p \in (1, \infty)$ . If  $u \in W_0^{1,p}(\Omega)$  then*

$$(1-s)[u]_{\mathcal{W}^{s,p}(\Omega)}^p \rightarrow K_{n,p} \int_{\Omega} |\nabla_x u|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y u|^p dx dy$$

as  $s \rightarrow 1^-$ .

*Proof.* By Lemma 4.3, we only need to proof that if  $u \in W_0^{1,p}(\Omega)$  then

$$(1-s) \left( [u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p - [u]_{\mathcal{W}^{s,p}(\Omega)}^p \right) \rightarrow 0$$

as  $s \rightarrow 1^-$ . First we prove the result for  $\phi \in C_0^\infty(\Omega)$ . We have

$$(4.10) \quad \begin{aligned} & \left( [\phi]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p - [\phi]_{\mathcal{W}^{s,p}(\Omega)}^p \right) = 2 \int_{\text{supp}(\phi)} \int_{\Omega_y^c} \frac{|\phi(x,y)|^p}{|x-z|^{n+sp}} dz dx dy \\ & \quad + 2 \int_{\text{supp}(\phi)} \int_{\Omega_x^c} \frac{|\phi(x,y)|^p}{|y-w|^{m+sp}} dw dx dy. \end{aligned}$$

Since  $\text{supp}(\phi) \subset \Omega$  is compact, there exists  $\delta > 0$  such that  $|x-z| > \delta$  and  $|y-w| > \delta$  for all  $(x,y) \in \text{supp}(\phi)$ ,  $z \in \Omega_y^c$ ,  $w \in \Omega_x^c$ . Then

$$\begin{aligned} & \int_{\text{supp}(\phi)} \int_{\Omega_y^c} \frac{|\phi(x,y)|^p}{|x-z|^{n+sp}} dz dx dy \leq \frac{\omega_n}{sp\delta^{sp}} \int_{\Omega} |\phi(x,y)|^p dx dy, \\ & \int_{\text{supp}(\phi)} \int_{\Omega_x^c} \frac{|\phi(x,y)|^p}{|y-w|^{m+sp}} dw dx dy \leq \frac{\omega_m}{sp\delta^{sp}} \int_{\Omega} |\phi(x,y)|^p dx dy. \end{aligned}$$

Therefore, using (4.10), we have that

$$(1-s) \left( [\phi]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p - [\phi]_{\mathcal{W}^{s,p}(\Omega)}^p \right) \rightarrow 0$$

as  $s \rightarrow 1^-$ .

The argument for the general case is analogous to the one performed in case 2 in the proof of Lemma 4.3.  $\square$

For the proof of the following lemma, see [11, Lemma 2].

**Lemma 4.5.** *Let  $\delta > 0$  and  $g, h: (0, \delta) \rightarrow (0, +\infty)$ . Assume that  $g(t) \leq g(t/2)$  and that  $h$  is non-increasing. Then*

$$\int_0^\delta t^{N-1} g(t) h(t) dt \geq \frac{N}{(2\delta)^N} \int_0^\delta t^{N-1} g(t) dt \int_0^\delta t^{N-1} h(t) dt$$

for all  $N > 0$ .

**Lemma 4.6.** *Let  $0 < s_0 < s$  and  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ . Then*

$$\frac{(1-s_0)[u]_{\mathcal{W}^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p} \text{diam}(\Omega)^{(s-s_0)p}} \leq (1-s)[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p$$

*Proof.* Let  $B_1$  and  $B_2$  be two balls in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively such that  $\Omega \subset B_1 \times B_2$  and  $\text{diam}(B_1) = \text{diam}(B_2) = \text{diam}(\Omega)$ . Then

$$\begin{aligned} & \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \geq \\ & \geq \int_{\mathbb{R}^m} \int_0^\infty \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{|u(x + tw, y) - u(x, y)|^p}{t^{1+sp}} dx d\sigma dt dy \\ & \geq \int_{\mathbb{R}^m} \int_0^{\text{diam}(\Omega)} \int_{S^{n-1}} t^{(1-s_0)p-1} \int_{\mathbb{R}^n} \frac{|u(x + t\omega, y) - u(x, y)|^p}{t^p} \frac{dx d\sigma dt dy}{t^{(s-s_0)p}} \end{aligned}$$

Taking  $N = (1-s_0)p$ ,  $\delta = \text{diam}(\Omega)$ , we get

$$g(t) = \int_{S^{n-1}} \int_{\mathbb{R}^m} \frac{|u(x + t\omega, y) - u(x, y)|^p}{t^p} dx d\sigma, \quad \text{and} \quad h(t) = \frac{1-s}{t^{(s-s_0)p}}.$$

By Lemma 4.5, we have that

$$\begin{aligned} & (1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+sp}} dz dx dy \geq \\ & \geq \frac{(1-s_0)p}{2^{(1-s_0)p} \text{diam}(\Omega)^{(1-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta t^{(1-s_0)p-1} g(t) dt \int_0^\delta t^{(1-s_0)p-1} h(t) dt \\ & \geq \frac{(1-s_0)p}{2^{(1-s_0)p} \text{diam}(\Omega)^{(1-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta t^{(1-s_0)p-1} g(t) dt \int_0^\delta (1-s) t^{(1-s)p-1} dt \\ & \geq \frac{(1-s_0)}{2^{(1-s_0)p} \text{diam}(\Omega)^{(s-s_0)p}} \int_{\mathbb{R}^m} \int_0^\delta \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{|u(x + t\omega, y) - u(x, y)|^p}{t^{1+s_0p}} dx d\sigma dt dy \\ & \geq \frac{(1-s_0)}{2^{(1-s_0)p} \text{diam}(\Omega)^{(s-s_0)p}} \int_{\Omega} \int_{\Omega_y} \frac{|u(x, y) - u(z, y)|^p}{|x - z|^{n+s_0p}} dz dx dy. \end{aligned}$$

Similarly

$$\begin{aligned} & (1-s) \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^n} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+sp}} dw dx dy \geq \\ & \geq \frac{(1-s_0)}{2^{(1-s_0)p} \text{diam}(\Omega)^{(s-s_0)p}} \int_{\Omega} \int_{\Omega_x} \frac{|u(x, y) - u(x, w)|^p}{|y - w|^{m+s_0p}} dw dx dy. \end{aligned}$$

This concludes the proof.  $\square$

We can now show the main result of this section.

**Theorem 4.7.** *Let  $\Omega$  is bounded domain in  $\mathbb{R}^{n+m}$  with smooth boundary,  $s \in (0, 1)$  and  $p \in (1, \infty)$ . Then*

$$\lim_{s \rightarrow 1^-} (1-s)\lambda_1(s, p) = \lambda_1(1, p).$$

*Proof.* First, we observe that, from Lemma 4.1, if  $u \in W_0^{1,p}(\Omega)$  then  $u \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$ . Then

$$(1-s)\lambda_1(s, p) \leq \frac{[u]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}^p}{\|u\|_{L^p(\Omega)}^p}$$

for all  $u \in W_0^{1,p}(\Omega)$ ,  $u \not\equiv 0$ . Therefore, by Lemma 4.3, we have that

$$\limsup_{s \rightarrow 1^-} (1-s)\lambda_1(s, p) \leq \frac{K_{n,p} \int_{\Omega} |\nabla_x u(x, y)|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y u(x, y)|^p dx dy}{\|u\|_{L^p(\Omega)}^p}$$

for all  $u \in W_0^{1,p}(\Omega)$ ,  $u \not\equiv 0$ . Then

$$(4.11) \quad \limsup_{s \rightarrow 1^-} (1-s)\lambda_1(s, p) \leq \lambda_1(1, p).$$

To finish the proof, we have to show that

$$\liminf_{s \rightarrow 1^-} (1-s)\lambda_1(s, p) \geq \lambda_1(1, p).$$

Let  $\{s_k\}_{k \in \mathbb{N}} \subset (0, 1)$  be such that  $s_k \rightarrow 1$  as  $k \rightarrow \infty$ ,

$$(4.12) \quad \lim_{k \rightarrow \infty} (1-s_k)\lambda_1(s_k, p) = \liminf_{s \rightarrow 1^-} (1-s)\lambda_1(s, p).$$

For each  $k \in \mathbb{N}$ , we let  $u_k$  be an eigenfunction corresponding to  $\lambda_1(s_k, p)$  such that  $\|u_k\|_{L^p(\Omega)} = 1$ . By (4.12), there is a positive constant  $C$  such that

$$(1-s_k)[u_k]_{\mathcal{W}^{s_k,p}(\mathbb{R}^{n+m})}^p \leq C \quad \forall k \in \mathbb{N}.$$

Then, by Lemma 2.4, there is a positive constant  $C$  such that

$$(1-s_k)\|u_k\|_{W^{s_k,p}(\mathbb{R}^{n+m})}^p \leq C \quad \forall k \in \mathbb{N}.$$

Thus, by [11, Corollary 7], up to a subsequence,  $\{u_k\}_{k \in \mathbb{N}}$  converges in  $L^p(\Omega)$  to some  $u \in W_0^{1,p}(\Omega)$ . Moreover, for all  $\delta > 0$ ,  $u_k \rightarrow u$  strongly in  $W^{1-\delta,p}(\Omega)$ . Therefore  $\|u\|_{L^p(\Omega)} = 1$ .

Let  $s_0 \in (0, 1)$ . Since  $s_k \rightarrow 1$ , there exists  $k_0 \in \mathbb{N}$  such that  $s_0 < s_k$  for all  $k \geq k_0$ . Then, by Lemma 4.6, we have that

$$\begin{aligned} \frac{(1-s_0)[u_k]_{\mathcal{W}^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p}} &\leq \text{diam}(\Omega)^{(s_k-s_0)p} (1-s_k)[u_k]_{\mathcal{W}^{s_k,p}(\mathbb{R}^n)}^p \\ &= \text{diam}(\Omega)^{(s_k-s_0)p} (1-s_k)\lambda_1(s_k, p). \end{aligned}$$

Thus, by (4.12) and Fatou's lemma, we get

$$\frac{(1-s_0)[u]_{\mathcal{W}^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p}} \leq \text{diam}(\Omega)^{(1-s_0)p} \liminf_{s \rightarrow 1^-} (1-s)\lambda_1(s, p).$$

By Corollary 4.4, it holds that

$$\begin{aligned} K_{n,p} \int_{\Omega} |\nabla_x u(x, y)|^p dx dy + K_{m,p} \int_{\Omega} |\nabla_y u(x, y)|^p dx dy &= \lim_{s_0 \rightarrow 1^-} \frac{(1-s_0)[u]_{\mathcal{W}^{s_0,p}(\Omega)}^p}{2^{(1-s_0)p}} \\ &\leq \liminf_{s \rightarrow 1^-} (1-s)\lambda_1(s, p). \end{aligned}$$

Then

$$\lambda_1(1, p) \leq \liminf_{s \rightarrow 1^-} (1-s)\lambda_1(s, p).$$

Therefore, by (4.11),

$$\lambda_1(1, p) = \lim_{s \rightarrow 1^-} (1-s)\lambda_1(s, p),$$

as we wanted to prove.  $\square$

## 5. THE LIMIT AS $p \rightarrow \infty$

Now we want to pass to the limit as  $p \rightarrow \infty$  in the first eigenvalue  $\lambda_1(s, p)$ . Our goal now is to show that

$$[\lambda_1(s, p)]^{1/p} \rightarrow \Lambda_{\infty}(s)$$

where

$$\Lambda_{\infty}(s) = \inf \{ [u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} : u \in \mathcal{W}^{s,\infty}(\mathbb{R}^{n+m}), \|u\|_{L^{\infty}(\Omega)} = 1, u = 0 \text{ in } \Omega^c \}.$$

Observe that, by Arzela-Ascoli's theorem, the previous infimum is attained.

We first prove a geometric characterization of  $\Lambda_{\infty}(s)$ .

**Lemma 5.1.** *Let  $R_s = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x-z|^s + |y-w|^s)$ , then*

$$\Lambda_{\infty}(s) = \frac{1}{R_s}.$$

*Proof.* Let  $u \in \mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})$ , such that  $\|u\|_{L^{\infty}(\Omega)} = 1$ ,  $u = 0$  in  $\Omega^c$  and  $\Lambda_{\infty}(s) = [u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})}$ . Then, let  $(x_0, y_0) \in \Omega$  be such that  $u(x_0, y_0) = 1$ . If  $(z, w) \in \partial\Omega$  we have

$$|u(x_0, y_0) - u(z, y_0)| \leq \Lambda_{\infty}(s)|x_0 - z|^s$$

and

$$|u(z, y_0) - u(z, w)| \leq \Lambda_{\infty}(s)|y_0 - w|^s.$$

Then

$$|u(x_0, y_0) - u(z, w)| \leq \Lambda_{\infty}(s)(|x_0 - z|^s + |y_0 - w|^s).$$

Therefore,

$$1 \leq \Lambda_{\infty}(s) \min_{(z,w) \in \partial\Omega} (|x_0 - z|^s + |y_0 - w|^s),$$

and then, we get

$$(5.1) \quad \Lambda_{\infty}(s) \geq \frac{1}{\min_{(z,w) \in \partial\Omega} (|x_0 - z|^s + |y_0 - w|^s)} \geq \frac{1}{R_s}.$$

Now, we choose  $(x_0, y_0)$  that solves the geometric maximization problem

$$R_s = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x-z|^s + |y-w|^s),$$

and consider the function

$$u(x, y) = \left(1 - \frac{|x_0 - x|^s + |y_0 - y|^s}{R_s}\right)_+.$$

Observe that,  $\|u\|_{L^\infty(\Omega)} = 1$ . On the other hand, since for any  $s \in (0, 1]$

$$|a^s - b^s| \leq |a - b|^s \quad \forall a, b \in [0, \infty),$$

we have that  $[u]_{\mathcal{W}^{s,\infty}(\mathbb{R}^{n+m})} \leq 1/R_s$ . Hence, using this functions as a test function in the variational problem defining  $\Lambda_\infty(s)$  we get

$$(5.2) \quad \Lambda_\infty(s) \leq \frac{1}{R_s}.$$

From (5.1) and (5.2) we obtain the desired result.  $\square$

**Lemma 5.2.** *Let  $u_p$  be a positive eigenfunction for  $\lambda_1(s, p)$  normalized according to  $\|u_p\|_{L^p(\Omega)} = 1$ . Then, there exists a sequence  $p_j \rightarrow \infty$  such that*

$$u_j \rightarrow u$$

*uniformly in  $\mathbb{R}^N$ . This limit function  $u$  belongs to the space  $\mathcal{W}^{s,\infty}(\Omega)$  and is a solution to the variational problem*

$$\Lambda_\infty(s) = \min \{ [u]_{\mathcal{W}^{s,\infty}(\Omega)} : u \in \mathcal{W}^{s,\infty}(\Omega), \|u\|_{L^\infty(\Omega)} = 1, u = 0 \text{ on } \partial\Omega \}.$$

*In addition, it holds that*

$$[\lambda_1(s, p)]^{1/p} \rightarrow \Lambda_\infty(s).$$

*Proof.* Let  $\alpha > 1$  and

$$R_{s\alpha} = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x - z|^{s\alpha} + |y - w|^{s\alpha}).$$

We first claim that

$$(5.3) \quad \frac{(R_s)^\alpha}{2^{\alpha-1}} \leq R_{s\alpha}$$

for any  $\alpha > 1$ . To this end, let  $(x_0, y_0) \in \Omega$  such that

$$R_s = \min_{(z,w) \in \partial\Omega} (|x_0 - z|^s + |y_0 - w|^s).$$

Then for all  $(z, w) \in \partial\Omega$  we have

$$\begin{aligned} (R_s)^\alpha &\leq (|x_0 - z|^s + |y_0 - w|^s)^\alpha \leq 2^{\alpha-1} (|x_0 - z|^{s\alpha} + |y_0 - w|^{s\alpha}) \\ &\leq 2^{\alpha-1} R_{s\alpha}, \end{aligned}$$

that is, (5.3). On the other hand, it is clear that if  $s\alpha < 1$  we have that

$$u_\alpha(x, y) = \left(1 - \frac{|x - x_0|^{\alpha s} + |y - y_0|^{\alpha s}}{R_{s\alpha}}\right)_+$$

belongs to  $\widetilde{\mathcal{W}}^{s,p}(\Omega)$  for all  $p > 1$ . Then

$$(5.4) \quad (\lambda_1(s, p))^{1/p} \leq \frac{[u_\alpha]_{\mathcal{W}^{s,p}(\mathbb{R}^{n+m})}}{\|u_\alpha\|_{L^p(\Omega)}}$$

for all  $p > 1$  and  $1 < \alpha < 1/s$ . Therefore

$$\limsup_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p} \leq \frac{[u_\alpha]_{\mathcal{W}^{s,\infty}(\Omega)}}{\|u_\alpha\|_{L^\infty(\Omega)}} \quad \forall \alpha \in (1, 1/s).$$

Observe that if  $\alpha \in (1, 1/s)$ , by (5.3), we have

$$\frac{|u_\alpha(x, y) - u_\alpha(z, y)|}{|x - z|^s} \leq \frac{|x - z|^{(\alpha-1)s}}{R_{s\alpha}} \leq 2^{\alpha-1} \frac{\text{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha}$$

for all  $(x, y) \neq (z, y) \in \overline{\Omega}$ , and

$$\frac{|u_\alpha(x, y) - u_\alpha(x, w)|}{|y - w|^s} \leq \frac{|y - w|^{(\alpha-1)s}}{R_{s\alpha}} \leq 2^{\alpha-1} \frac{\text{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha},$$

for all  $(x, y) \neq (z, y) \in \overline{\Omega}$ , that is

$$[u_\alpha]_{\mathcal{W}^{s,\infty}(\Omega)} \leq 2^{\alpha-1} \frac{\text{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha}.$$

Then, by (5.4) we get

$$\limsup_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p} \leq 2^{\alpha-1} \frac{\text{diam}(\Omega)^{(\alpha-1)s}}{(R_s)^\alpha} \quad \alpha \in (1, 1/s),$$

since  $\|u_\alpha\|_{L^\infty(\Omega)} = 1$ . Therefore, passing to the limit as  $\alpha \rightarrow 1$  in the previous inequality we get

$$(5.5) \quad \limsup_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p} \leq \frac{1}{R_s} = \Lambda_\infty(s).$$

Our next goal is to show that

$$\Lambda_\infty(s) \leq \liminf_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p}.$$

Let  $p_j > 1$  be such that

$$\liminf_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p} = \lim_{j \rightarrow \infty} (\lambda_1(s, p_j))^{1/p_j}.$$

By (5.5), without of loss of generality, we can assume

$$(\lambda_1(s, p_j))^{1/p_j} = [u_{p_j}]_{\mathcal{W}^{s,p_j}(\mathbb{R}^{n+m})} \leq \Lambda_\infty(s) + \epsilon \quad \forall j \in \mathbb{N},$$

where  $u_{p_j}$  is an eigenfunction for  $\lambda_1(s, p_j)$  normalized according to  $\|u_{p_j}\|_{L^{p_j}(\Omega)} = 1$  and  $\epsilon$  is any positive number. Then, by Lemma 2.4, we have that there exists a constant  $C$ , independent of  $j$ , such that

$$|u_{p_j}|_{W^{s,p_j}(\Omega)} \leq C \quad \forall j \in \mathbb{N}.$$

Therefore, for any  $j \in \mathbb{N}$  there exists a constant  $C$  independent of  $j$ , such that

$$(5.6) \quad \|u_{p_j}\|_{W^{s,p_j}(\Omega)} \leq C.$$

On the other hand, given  $q > 1$  such that  $sq > 2(n+m)$  and taking  $t = s - n+m/q$ , by Hölder's inequality, for any  $p_j > q$  we have that

$$\|u_{p_j}\|_{L^q(\Omega)}^q \leq |\Omega|^{1-\frac{q}{p_j}} \|u_{p_j}\|_{L^{p_j}(\Omega)}^q = |\Omega|^{1-\frac{q}{p_j}},$$



and

$$\begin{aligned} |u_{p_j}|_{W^{t,q}(\Omega)}^q &= \int_{\Omega^2} \frac{|u_{p_j}(x,y) - u_{p_j}(z,w)|^q}{|(x,y) - (z,w)|^{sq}} dx dy dz dw \\ &\leq |\Omega|^{2(1-\frac{q}{p_j})} \left( \int_{\Omega^2} \frac{|u_{p_j}(x,y) - u_{p_j}(z,w)|^{p_j}}{|(x,y) - (z,w)|^{sp_j}} dx dy dz dw \right)^{\frac{q}{p_j}} \\ &\leq |\Omega|^{2(1-\frac{q}{p_j})} \max \left\{ 1, \text{diam}(\Omega)^{(n+m)\frac{q}{p_j}} \right\} |u_{p_j}|_{W^{s,p_j}(\Omega)}^q. \end{aligned}$$

Hence, by (5.6), for  $j$  large there exists a constant  $C$ , independent of  $j$ , such that

$$\|u_{p_j}\|_{W^{t,q}(\Omega)} \leq C \max \left\{ |\Omega|^{\frac{1}{q}-\frac{1}{p_j}}, |\Omega|^{2(\frac{1}{q}-\frac{1}{p_j})}, |\Omega|^{2(\frac{1}{q}-\frac{1}{p_j})} \text{diam}(\Omega)^{\frac{n+m}{p_j}} \right\},$$

that is, there exists  $j_0 > 1$  such that  $\{u_{p_j}\}_{j>j_0}$  is bounded in  $W^{t,q}(\Omega)$ . Then, since  $tq > n+m$ , by Theorem 2.3, there exists a subsequence  $\{u_k\}_{k \in \mathbb{N}}$  of  $\{u_{p_j}\}_{j>j_0}$  and a function  $u \in C^{0,\gamma}(\overline{\Omega})$  ( $0 < \gamma < t - (n+m)/q$ ) such that  $u_k \rightarrow u$  uniformly in  $\overline{\Omega}$ .

Thus, if  $q > 1$  there exists  $k_0 \in \mathbb{N}$  such that  $p_k > q$  if  $k > k_0$  and therefore, by Hölder's inequality, for any  $k > k_0$  we have

$$\begin{aligned} &\left( \int_{\Omega} \int_{\Omega_y} \frac{|u_k(x,y) - u_k(z,y)|^q}{|x-z|^{qs}} dz dx dy \right)^q \\ &\leq C^{\frac{1}{q}-\frac{1}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{n}{p_k}} \right\} \left( \int_{\Omega} \int_{\Omega_y} \frac{|u_k(x,y) - u_k(z,y)|^{p_k}}{|x-z|^{p_k s+n}} dz dx dy \right)^{\frac{1}{p_k}} \\ &\leq C^{\frac{1}{q}-\frac{q}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{n}{p_k}} \right\} [u_k]_{\mathcal{W}^{s,p_k}(\Omega)}, \end{aligned}$$

and similarly

$$\begin{aligned} &\left( \int_{\Omega} \int_{\Omega_x} \frac{|u_k(x,y) - u_k(x,w)|^q}{|y-w|^{qs}} dw dx dy \right)^q \\ &\leq C^{\frac{1}{q}-\frac{q}{p_k}} \max \left\{ 1, \text{diam}(\Omega)^{\frac{m}{p_k}} \right\} [u_k]_{\mathcal{W}^{s,p_k}(\Omega)}. \end{aligned}$$

Here  $C$  is a constant independent of  $k$ . Then passing to the limit as  $k \rightarrow \infty$  and using Fatou's lemma we have that

$$\begin{aligned} &\left( \int_{\Omega} \int_{\Omega_y} \frac{|u(x,y) - u(z,y)|^q}{|x-z|^{qs}} dz dx dy \right)^q \leq C^{\frac{1}{q}} \liminf_{k \rightarrow \infty} [u_k]_{\mathcal{W}^{s,p_k}(\Omega)} \\ &\leq C^{\frac{1}{q}} \liminf_{p \rightarrow \infty} (\lambda_1(s,p))^{1/p}, \\ &\left( \int_{\Omega} \int_{\Omega_x} \frac{|u(x,y) - u(x,w)|^q}{|y-w|^{qs}} dw dx dy \right)^q \leq C^{\frac{1}{q}} \liminf_{k \rightarrow \infty} [u_k]_{\mathcal{W}^{s,p_k}(\Omega)} \\ &\leq C^{\frac{1}{q}} \liminf_{p \rightarrow \infty} (\lambda_1(s,p))^{1/p} \end{aligned}$$

for all  $q > 1$ . Now passing to the limit as  $q \rightarrow \infty$  we obtain

$$\begin{aligned} \sup \left\{ \frac{|u(x,y) - u(z,y)|}{|x-z|^s} : (x,y) \neq (z,y) \in \Omega \right\} &\leq \liminf_{p \rightarrow \infty} (\lambda_1(s,p))^{1/p}, \\ \sup \left\{ \frac{|u(x,y) - u(x,w)|}{|x-z|^s} : (x,y) \neq (x,w) \in \Omega \right\} &\leq \liminf_{p \rightarrow \infty} (\lambda_1(s,p))^{1/p}, \end{aligned}$$

that is

$$(5.7) \quad [u]_{\mathcal{W}^{s,\infty}(\Omega)} \leq \liminf_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p}.$$

To conclude we need to show that  $\|u\|_{L^\infty(\Omega)} = 1$ . For all  $q > 1$  there exists  $k_0 \in \mathbb{N}$  such that  $p_k > q$  if  $k > k_0$  and therefore, by Hölder's inequality, for any  $k > k_0$  we get

$$\|u_k\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{p_k}} \|u_{p_j}\|_{L^{p_j}(\Omega)}^q = |\Omega|^{\frac{1}{q} - \frac{1}{p_j}}.$$

Then passing to the limit as  $k \rightarrow \infty$  and using that  $u_k \rightarrow u$  uniformly in  $\overline{\Omega}$ ,  $\|u\|_{L^q(\Omega)} \leq 1$  for all  $q > 1$ . Hence  $\|u\|_{L^\infty(\Omega)} \leq 1$ . On the other hand, for all  $k$  we have  $1 = \|u_k\|_{L^{p_k}(\Omega)} \leq |\Omega|^{1/p_k} \|u_k\|_{L^\infty(\Omega)}$ . Then, since  $u_k \rightarrow u$  uniformly in  $\overline{\Omega}$ , we get  $1 \leq \|u\|_{L^\infty(\Omega)}$ . Hence  $\|u\|_{L^\infty(\Omega)} = 1$ . Thus, by (5.7), we get

$$\Lambda_\infty(s) \leq [u]_{\mathcal{W}^{s,\infty}(\Omega)} \leq \liminf_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p},$$

and by (5.5) we conclude that

$$\Lambda_\infty(s) = \lim_{p \rightarrow \infty} (\lambda_1(s, p))^{1/p}.$$

This ends the proof.  $\square$

Using the geometric characterization given in Lemma 5.1 we can compute  $\Lambda_\infty(s)$  in some concrete examples.

**Example 1.** When  $\Omega = B_R$  is a ball of radius  $R$  we have

$$\Lambda_\infty(s) = \frac{1}{R^s}.$$

**Example 2.** When  $\Omega = (-R, R) \times (-L, L)$  is a rectangle in  $\mathbb{R}^2$  we have

$$\Lambda_\infty(s) = \frac{1}{\min\{R^s, L^s\}}.$$

*Remark 5.3.* One can consider two different powers  $r$  and  $s$  in the definition of the pseudo  $p$ -Laplacian. In this case we get that,

$$\Lambda_\infty(r, s) = \max_{(x,y) \in \Omega} \min_{(z,w) \in \partial\Omega} (|x - z|^r + |y - w|^s).$$

**Viscosity solutions.** To obtain an eigenvalue problem that is satisfied by the limit of the eigenfunctions  $u_p$  when  $p \rightarrow \infty$ , we need to introduce the definition of viscosity solutions. This is a notion of solution different from the weak one considered before. We refer to [13] for an introduction to the subject of viscosity solutions. In the theory of viscosity solutions the equation is evaluated for test functions at points where they touch the graph of a solution. Viscosity solutions are assumed to be continuous and the fractional Sobolev space is absent from the definition (no derivatives of a solutions are needed).

**Definition 5.4.** (Viscosity solutions). Suppose that the function  $u$  is continuous in  $\mathbb{R}^{n+m}$  and that  $u = 0$  in  $\Omega^c$ . We say that  $u$  is a viscosity supersolution of the equation  $-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0$  if the following holds: whenever  $x_0 \in \Omega$  and  $\varphi \in C_0^1(\mathbb{R}^{n+m})$  (the test function) are such that  $\varphi(x_0) = u(x_0)$  and  $\varphi(x) \leq u(x)$  for every  $x \in \mathbb{R}^{n+m}$ , then we have

$$-\mathcal{L}_{s,p}\varphi(x_0) + \lambda|\varphi(x_0)|^{p-2}\varphi(x_0) \leq 0.$$

The requirement for being a viscosity subsolution is symmetric: the test function is touching from above and the inequality is reversed.

Finally, a viscosity solution is defined as being both a viscosity supersolution and a viscosity subsolution.

For our eigenvalue problem, we have that a continuous weak solution is a viscosity solution. For the proof we refer to [29].

**Theorem 5.5.** *An eigenfunction  $u \in C(\overline{\Omega})$  (in the weak sense) is a viscosity solution of the equation  $-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0$  in the sense of Definition 5.4.*

We will also use the following lemmas.

**Lemma 5.6.** *Assume that*

$$\begin{aligned} (A_p)^{1/p} &\rightarrow A, & (B_p)^{1/p} &\rightarrow -B, \\ (C_p)^{1/p} &\rightarrow C, & (D_p)^{1/p} &\rightarrow -D, \end{aligned}$$

and that

$$\theta_p \rightarrow \Theta,$$

as  $p \rightarrow \infty$ . If

$$2^{1/p}(A_p + C_p)^{1/p} \geq (B_p + D_p + \theta_p^{p-1})^{1/p}$$

for every  $p$  large enough, then, passing to the limit, it holds that

$$\max\{A; C\} \geq \max\{-B; -D; \Theta\}.$$

*Proof.* First, assume that  $A > C$  and  $-B > \max\{-D; \Theta\}$ . Then for  $p$  large enough we have  $A_p \geq C_p$ ,  $-B_p \geq -D_p$  and  $-B_p \geq (\theta_p)^p$ . Then taking  $p \rightarrow \infty$  in

$$(A_p)^{1/p} 2^{1/p} \left(1 + \frac{C_p}{A_p}\right)^{1/p} \geq (B_p)^{1/p} \left(1 + \frac{D_p}{B_p} + \frac{\theta_p^{p-1}}{B_p}\right)^{1/p}$$

we get

$$A \geq -B.$$

The rest of the cases ( $A = C$ ,  $A < C$ , etc) can be handled in an analogous way.  $\square$

**Lemma 5.7.** *For a smooth test function  $\phi$  let*

$$A_p = \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz.$$

If  $x_p \rightarrow x_0$ ,  $y_p \rightarrow y_0$  as  $p \rightarrow \infty$ , then

$$(A_p)^{1/p} \rightarrow A = \sup_z \frac{\phi(x_0, y_0) - \phi(z, y_0)}{|x_0 - z|^s}.$$

*Proof.* We just have to observe that

$$(A_p)^{1/p} = \left( \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz \right)^{1/p}.$$

The integrand satisfies

$$\begin{aligned} & \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} \\ & \sim \frac{|\phi(x_0, y_0) - \phi(z, y_0)|^{p-2} (\phi(x_0, y_0) - \phi(z, y_0))^+}{|x_0 - z|^{n+sp}} \end{aligned}$$

and hence the result follows from the fact that  $(\int f^p)^{1/p} \rightarrow \|f\|_\infty$ .  $\square$

**Lemma 5.8.** *Any uniform limit of  $u_p$  a sequence of eigenfunctions for  $\lambda_1(s, p)$  normalized according to  $\|u_p\|_{L^p(\Omega)} = 1$ ,  $u$  is a nontrivial solution to*

$$\begin{cases} \max\{A; C\} = \max\{-B; -D; \Lambda_\infty(s)u\} & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

in the viscosity sense. Here

$$\begin{aligned} A &= \sup_w \frac{u(x, w) - u(x, y)}{|y - w|^s}, & B &= \inf_w \frac{u(x, w) - u(x, y)}{|y - w|^s}, \\ C &= \sup_z \frac{u(z, y) - u(x, y)}{|x - z|^s}, & D &= \inf_z \frac{u(z, y) - u(x, y)}{|x - z|^s}. \end{aligned}$$

*Proof.* We call  $u_p$  a sequence of solutions to  $-\mathcal{L}_{s,p}u + \lambda|u|^{p-2}u = 0$  that converges uniformly to  $u$ . That  $u = 0$  in  $\Omega^c$  follows since  $u_p = 0$  in  $\Omega^c$  and we have uniform convergence.

Let  $\phi \in C_0^1(\mathbb{R}^{n+m})$  be such that  $u - \phi$  has a strict minimum at  $(x_0, y_0) \in \Omega$ . Since  $u_p$  converges uniformly to  $u$  we have that there exist  $(x_p, y_p) \in \Omega$  such that  $u_p - \phi$  has a minimum at  $(x_p, y_p)$  and  $(x_p, y_p) \rightarrow (x_0, y_0)$  as  $p \rightarrow \infty$ . Since  $u_p$  is a viscosity solution to  $-\mathcal{L}_{s,p}v(x, y) + \lambda_1(s, p)v(x, y)^{p-1} = 0$  in  $\Omega$ , we obtain

$$\begin{aligned} & ((\lambda_1(s, p))^{1/(p-1)} u_p(x_p, y_p))^{p-1} \leq \\ & \leq 2 \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))}{|x_p - z|^{n+sp}} dz \\ & + 2 \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2} (\phi(x_p, y_p) - \phi(x_p, w))}{|y_p - w|^{m+sp}} dw \\ & = 2(A_p - B_p + C_p - D_p), \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} A_p &= \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))^+}{|x_p - z|^{n+sp}} dz, \\ B_p &= \int_{\mathbb{R}^n} \frac{|\phi(x_p, y_p) - \phi(z, y_p)|^{p-2} (\phi(x_p, y_p) - \phi(z, y_p))^-}{|x_p - z|^{n+sp}} dz, \\ C_p &= \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2} (\phi(x_p, y_p) - \phi(x_p, w))^+}{|y_p - w|^{m+sp}} dw, \\ D_p &= \int_{\mathbb{R}^m} \frac{|\phi(x_p, y_p) - \phi(x_p, w)|^{p-2} (\phi(x_p, y_p) - \phi(x_p, w))^-}{|y_p - w|^{m+sp}} dw. \end{aligned}$$

We observe that

$$\begin{aligned} (A_p)^{1/p} &\rightarrow A, & (B_p)^{1/p} &\rightarrow -B, \\ (C_p)^{1/p} &\rightarrow C, & (D_p)^{1/p} &\rightarrow -D, \end{aligned}$$

and

$$(\lambda_1(s, p))^{1/(p-1)} u_p(x_p, y_p) \rightarrow \Lambda_\infty u(x_0, y_0).$$

Hence, taking limit as  $p \rightarrow \infty$  in (5.8), from Lemma 5.6, we get

$$\max\{-B; -D; \Lambda_\infty(s)u(x_0, y_0)\} \leq \max\{A; C\}.$$

Now, if  $\psi$  is such that  $u - \psi$  has a strict minimum at  $(x_0, y_0) \in \Omega$ . Since  $u_p$  converges uniformly to  $u$  we have that there exist  $(x_p, y_p) \in \Omega$  such that  $u_p - \psi$  has a minimum at  $(x_p, y_p)$  and  $(x_p, y_p) \rightarrow (x_0, y_0)$  as  $p \rightarrow \infty$ . Since  $u_p$  is a solution to  $-\mathcal{L}_{s,p}v(x, y) + \lambda v(x, y)^{p-1} = 0$  in  $\Omega$  we obtain

$$\begin{aligned} &((\lambda_{1,p})^{1/(p-1)} u_p(x_p, y_p))^{p-1} \geq \\ &\geq 2 \int_{\mathbb{R}^n} \frac{|\psi(x_p, y_p) - \psi(z, y_p)|^{p-2} (\psi(x_p, y_p) - \psi(z, y_p))}{|x_p - z|^{n+sp}} dz \\ &+ 2 \int_{\mathbb{R}^m} \frac{|\psi(x_p, y_p) - \psi(x_p, w)|^{p-2} (\psi(x_p, y_p) - \psi(x_p, w))}{|y_p - w|^{m+sp}} dw, \end{aligned}$$

and, arguing as before, we obtain

$$\max\{A; C\} \geq \max\{-B; -D; \Lambda_\infty(s)u(x_0, y_0)\}.$$

□

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